FOM 12

5.7 – Expected Value

The **Expected Value** of an "experiment" is the long-run average. If the experiment could be repeated many times, the expected value is the average of all the results. (All Outcomes)

$$Expected\ Value = \begin{pmatrix} probability \\ of\ event\ 1 \end{pmatrix}\begin{pmatrix} payoff\ for \\ event\ 1 \end{pmatrix} + \begin{pmatrix} probability \\ of\ event\ 2 \end{pmatrix}\begin{pmatrix} payoff\ for \\ event\ 2 \end{pmatrix} + \cdots$$

Example 1: Clark Griswold pays \$1.75 to play a game at the Klondike. He picks one card from a standard deck of 52 playing cards. If the card is Red he wins \$1.50.
What is the expected value for this game?

If card isn't red Clark loses \$1.75 if It is red he still loses \$0.25 (Paid \$1.75 to play & won \$1.50 back)

Expected Value = P(Pick Red) x \$-0.25

$E(\text{Profit}) = \frac{26}{52} \times -0.25 = -\frac{6.5}{52} = -0.125$

$E(\text{Profit}) = \$-0.13$ Clark will lose \$0.13 each game (on average if he plays continuously)

Example 2: A radio station is offering a contest. There are 4 containers each with 10 balls numbered 0 through 9. To play a contestant chooses a 4-digit number and pays \$1. If the contestant's number is drawn, they win \$5000. If the number is not drawn, the contestant loses their dollar. What is the expected value for this game?

Possible Results: Contestant's Number Drawn & Number Not Drawn.

Expected Value: P(Number Drawn) x \$4999 (still paid \$1) + P(Not Drawn) x \$-1.00

(Number of Possible ways to draw the 4 digits: $10 \times 10 \times 10 \times 10 = 10000$)

$E(\text{Profit}) = \frac{1}{10000} \times 4999 + \left(1 - \frac{1}{10000}\right) \times -1 = \frac{4999}{10000} - \frac{9999}{10000} = \frac{-5000}{10000}$

$E(\text{Profit}) = \$-0.50 = \$-0.50$ Expected Value of game is \$-0.50

Example 3: You draw one card from a standard deck of playing cards. If you pick a spade you will win \$5. If you pick a red face card you win \$8. If you pick any other card, you lose \$4. Do you want to play? Explain. Results: Spade & Red Face Card & Neither

Expected Value: P(spade) x \$5 + P(Red Face) x \$8 + P(Neither) x \$-4

$EV = \frac{13}{52} \times 5 + \frac{6}{52} \times 8 + \frac{33}{52} \times -4 = \frac{65}{52} + \frac{48}{52} - \frac{132}{52} = \frac{-19}{52}$

Expected Value = \$-0.365384615

The expected value is \$-0.37

What you pay each year for insurance.

Example 4: You take out a fire insurance policy on your home. The annual premium is \$300. In case of fire, the insurance company will pay you \$200,000. The probability of a house fire in your area is 0.0002.

a) What is the expected value? Possible Results: No Fire & Fire.

Expected Value = P(No Fire) × (−\$300) + P(Fire) × (\$199700)

you still paid \$300

$EV = (1-0.0002)(-300) + (0.0002)(199700)$

$= (0.9998)(-300) + (0.0002)(199700)$

$= -299.94 + 39.94$

$= -260$

The expected value is \$−260

The customer still paid them \$300

b) What is the insurance company's expected value?

Expected Value = P(No Fire) × (\$300) + P(Fire) × (−199700)

$EV = (1-0.0002)(300) + (0.0002)(-199700)$

$= (0.9998)(300) + (0.0002)(-199700)$

$= 299.94 - 39.94$

$= 260$

The Insurance Company's expected value is \$260

c) Suppose the insurance company sells 100,000 of these policies. What can the company expect to earn?

For each policy they sell they expect to keep \$260 of the premium.

so 100000 policies × 260 = 26,000,000

The company will earn \$26,000,000

Example 5: You and your friend, Jeremy, are fishing in a pond that contains 10 trout and 10 sunfish. Each time one of you catches a fish, you release it back into the water. Jeremy offers you the choice of 2 different bets.

Bet 1: If the next 3 fish he catches are all sunfish, you will pay him \$100, otherwise he will pay you \$20.

Bet 2: If you catch at least 2 sunfish of the next 3 fish that you catch, he will pay you \$50, otherwise you will pay him \$25.

a) What is the expected value from bet 1?

Possible Outcomes: Next 3 Sunfish & Not all 3 are Sunfish

Expected Value = P(3 sunfish) x \$-100 + P(Not 3 sunfish) x \$20

$P(3\text{ Sunfish}) = \frac{10}{20} \times \frac{10}{20} \times \frac{10}{20} = \frac{1000}{8000} = \frac{1}{8}$ $\quad P(\text{Not 3 Sunfish}) = \left(1-\frac{1}{8}\right) = \frac{7}{8}$

$EV = \left(\frac{1}{8}\right)\times(-100) + \left(\frac{7}{8}\right)(20) = \frac{-100}{8} + \frac{140}{8} = \frac{40}{8} = 5$

$\boxed{\text{The expected value is \$5}}$

b) What is the expected value from bet 2?

Possible Outcomes: At least 2 sunfish & less than 2 sunfish

SAMPLE SPACE (P(T) = P(S))

TTT TSS
TTS STS
TST SST
STT SSS

8 Outcomes ⇒ 4 = at least 2
4 = less than 2.

Expected Value = P(At least 2 S)(x\$50) + P(less 2) x (-\$25)

$EV = \frac{4}{8} \times 50 + \frac{4}{8} \times -25 = \frac{200}{8} - \frac{100}{8}$

$= \frac{100}{8} = 12.5$

$\boxed{\text{The expected value is \$12.50}}$

c) your friend says he is willing to take both bets a combined total of 50 times. If you want to maximize your expected value, what should you do?

A. Take bet 1 all 50 times

B. Take bet 2 all 50 times

C. Take bet 1 twenty times and bet 2 thirty times

D. Take neither bet

In both bets the expected value is in your favour. (what was your friend thinking?) But bet #2 has a higher expected value, so take bet 2 every time!

Example 6: Ahmed is playing a lottery game where he must pick 6 numbers from the numbers 1-49. If his ticket matches all 6 numbers (in any order), he wins the grand prize of $6 million. If only 5 of his numbers match he wins $2000.00. The game costs $3. What is the expected value from playing this lottery game?

Possible Results (Events): Match all 6, Match 5, Neither.

How many ways can the 6 numbers be chosen? (order not important)

$${}_{49}C_6 = 13983816$$

$$P(\text{Match } 6) = \frac{1}{13983816}$$

only 5 of your 6 match

Your 6th number must be one of the 43 #s that were not drawn

$$P(\text{Match } 5) = \frac{{}_6C_5 \times {}_{43}C_1}{13983816} = \frac{6 \times 43}{13983816} = \frac{258}{13983816}$$

$$P(\text{No Win}) = 1 - P(\text{Match } 6 \text{ OR Match } 5)$$

$$= 1 - \left(\frac{1}{13983816} + \frac{258}{13983816}\right) = 1 - \frac{259}{13983816}$$

$$= \frac{13983816}{13983816} - \frac{259}{13983816} = \frac{13983557}{13983816}$$

Expected Value $= P(\text{Match } 6) \times 5999997 + P(\text{Match5}) \times 1997 + P(\text{No Win}) \times -3$

The game costs $3 (even if you win)

$$EV = \frac{1}{13983816} \times 5999997 + \frac{258}{13983816} \times 1997 + \frac{13983557}{13983816} \times -3$$

$$= \frac{5999997}{13983816} + \frac{515226}{13983816} - \frac{41950671}{13983816}$$

$$= \frac{-35435448}{13983816} = -2.53403277$$

$$= \$-2.53$$

Assignment: Expected Value Worksheet

$\boxed{\text{The expected value is } \$-2.53}$